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Finite-Difference Method for Generalized Eigenvalue Problem in Ordinary Differential Equations

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A numerical solution of the generalized eigenvalue problem for a system of linear ordinary differential equation using finite difference method is considered. It is shown how the zeroth order numerical approximation can be improved to give higher order accuracy. Further a scheme for locating all the eigenvalues in any given finite region in the complex plane is discussed.

1. INTRODUCTION

We consider the generalized eigenvalue problem for a general system of m first-order linear homogeneous ordinary differential equations

$$B(t, \lambda) \frac{dy(t)}{dt} = C(t, \lambda)\, y(t) \tag{1}$$

subject to m linearly independent linear homogeneous boundary conditions of the form

$$B_a(\lambda)\, y(a) = 0, \qquad B_b(\lambda)\, y(b) = 0, \tag{2}$$

where $y(t)$ is an m-vector, $B(t, \lambda)$ and $C(t, \lambda)$ are $m \times m$ matrices. $B_a(\lambda)$ is an $m_l \times m$ matrix of rank m_l, while $B_b(\lambda)$ is an $(m - m_l) \times m$ matrix of rank $m - m_l$, $t = a$ and $t = b$ are the two end points, and we are looking for the solution in the interval $[a, b]$. Here λ is a parameter to be determined in such a manner that the nontrivial solution of Eq. (1) satisfies the boundary conditions (2). All the functions involved are assumed to be continuous and differentiable to any order required. Further matrices, B, C, B_a, and B_b are assumed to be polynomials in λ of degree m_1, m_2, m_3, and m_4, respectively, and $r = \mathrm{Max}(m_1, m_2, m_3, m_4)$. For example,

$$B(t, \lambda) = B_0(t) + \lambda B_1(t) + \cdots + \lambda^{m_1} B_{m_1}(t),$$

where all B_i's are $m \times m$ matrices independent of λ.

There are two types of simple methods for numerical solution of such problems. The first type based on the solution of initial value problem has been discussed by

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Conte [1] and Antar [2], while the second type is the finite-difference method discussed by Fox [3], Gary [4], and Keller [5]. In this paper we shall consider only the finite-difference methods by which the continuous eigenvalue problem may be transformed into a discrete problem.

The finite-difference method suffers from two drawbacks. The first is that it requires an excessively large number of mesh points to get a reasonable accuracy, since the truncation errors involved are of the order of h^2 or $1/N^2$, where N is the number of intervals in the finite-difference scheme. To overcome this difficulty the use of more accurate difference schemes or the Richardson's $h \to 0$ extrapolation has been used (cf. Keller [5]). In this work we have used the explicit calculation of first-order difference corrections giving an accuracy of the order of h^4 or $1/N^4$, which is sufficient for most purposes. This procedure has the advantage that very few extra calculations are required to obtain the first-order correction.

The second difficulty arises from the iterative methods which are normally employed to determine the zeros of a determinant. These methods work well only if the approximate location of zeros is known in advance. To overcome this problem we have used a method discussed by Delves and Lyness [6], which can determine all the eigenvalues in any given finite region of complex plane and does not require any previous knowledge about the eigenvalues.

The generalized eigenvalue problem can arise in the solution of ordinary or partial differential equations especially when the Fourier transform is used. One comes across such problems in the study of linear stability analysis in the area of hydrodynamics. For example, for a polytropic fluid layer bounded by plane horizontal boundaries in the presence of uniform gravitational field acting in the z-direction if the infinitesimal pertubations are assumed to be of the form $f(z)\exp(\lambda t + ikx)$, the linearized equations for optically thin disturbances come out to be (cf. Antia *et al.* [8])

$$T_0\lambda(\gamma\lambda + q)\frac{d(\rho_0 W)}{dz} = \lambda^2\left(1 - \frac{\gamma}{\Gamma}\right)(\rho_0 W) - [\lambda^2(\lambda + q) + k^2\, T_0(\gamma\lambda + q)]\, P_1\,,$$

$$T_0(\gamma\lambda + q)\frac{dP_1}{dz} = \left[\left(1 - \frac{\gamma}{\Gamma}\right) - \lambda T_0(\gamma\lambda + q)\right](\rho_0 W) - (\lambda + q)\, P_1\,, \qquad (3)$$

where P_0, ρ_0, and T_0 are respectively the pressure, density, and temperature in the zeroth-order steady state which are functions of z and P_1, ρ_1, and T_1 are infinitesimal perturbations to these quantities, W is the velocity in vertical direction, and $\gamma = C_p/C_v$ is the ratio of specific heats. The unperturbed quantities for the polytropic fluid with polytropic index Γ are given by

$$T_0 = 1 - \frac{\Gamma - 1}{\Gamma} z, \qquad P = \rho_0{}^{\Gamma}. \qquad (4)$$

We shall discuss the numerical solution of this problem in Section 4.

2. The Difference Approximation

The differential equation can be written in the form

$$L(\lambda)\, y(t) \equiv B(t, \lambda) \frac{dy(t)}{dt} - C(t, \lambda)\, y(t) = 0. \tag{5}$$

In the interval $[a, b]$ we define a uniform mesh

$$t_j = a + jh, \qquad j = 0, 1, 2, \ldots, N, \quad h = (b - a)/N.$$

At each point t_j of the mesh we seek a vector u_j which is to approximate $y(t_j)$, the solution of (1)-(2). The approximating net function $\{u_j\}$ is determined by the corresponding difference equation $L_h u_j = 0$. The central difference scheme is the simplest in which we approximate the differential equation by

$$\begin{aligned} hL_h(\lambda)\, y(t_{j-1/2}) &= B(t_{j-1/2}, \lambda)(u_j - u_{j-1}) \\ &\quad - \tfrac{1}{2}hC(t_{j-1/2}, \lambda)(u_j + u_{j-1}) \end{aligned} \tag{6}$$

where

$$t_{j-1/2} = t_j - \tfrac{1}{2}h.$$

This provides mN equations in $m(N + 1)$ unknowns formed by components of u_j $(j = 0, 1, 2, \ldots, N)$. The remaining m equations are supplied by the boundary conditions

$$B_a(\lambda)\, u_0 = 0, \qquad B_b(\lambda)\, u_N = 0. \tag{7}$$

The system of Eqs. (6) and (7) can be written in the matrix form as $AX = 0$, by suitably defining the $m(N + 1) \times m(N + 1)$ matrix A.

Now the problem has been reduced to solving a system of $(N + 1)$ m linear homogeneous algebraic equations in equal number of unknowns. Thus for nontrivial solutions to exist we must have $\det(A) = 0$, and the zeros of $\det(A)$ will give the desired eigenvalues. For this purpose any root-finding method such as the Muller's method or the secant iteration method can be used to determined the complex roots, while the real roots for which the determinant is also real can be located by looking for sign changes. In some cases, the successive minors of the determinant form a Sturm sequence which can be used to locate these roots very efficiently. The roots can then be accurately determined by appropriate coupling of methods of bisection and the secant iteration. The determinant for this purpose can be evaluated by using the usual Gaussian elimination method (cf. Wilkinson [7]). It is found that pivoting is not necessary in this process.

However, the iterative methods used suffer from a serious drawback as it is essential to know the eigenvalues approximately, otherwise it is impossible for the iteration to converge in a reasonable number of attempts. For the hydrodynamic stability analysis it turns out that in the absence of dissipation (e.g., $q = 0$ in Eq. (3)) the eigenvalues are real or purely imaginary, that is, λ^2 is always real. Such eigenvalues

can then be easily located by looking for sign changes since the determinant also can be written as a real function of λ^2. Further, since the dissipative effects are usually small, these eigenvalues serve as a fairly good approximation to the eigenvalues for the actual problem and can be very effectively used as starting values for the iterations. But in general it is not possible to get such approximations. For example, in the stability analysis of a polytropic fluid layer with a uniform horizontal magnetic field it is found that the "slow modes" cannot be computed by iterative methods even if a large number of iterations are carried out. In order to overcome such problems we have used a method described by Delves and Lyness [6], which is based on integration in the complex plane. This method can find out all the zeros of any analytic function in any given finite region of complex plane. For the present problem it can be easily seen that the determinant is an analytic function of λ. However, this method also requires the derivative of the determinant with respec to λ, which will clearly require a considerable amount of computation if it is directly evaluated and hence a numerical approximation to it is used.

We have adopted square contours and Simpson's one-third rule for evaluating the integrals. The derivative is approximated by using a five-point difference formula. Both of these processes give an accuracy of the order of h_1^4, where h_1 is the step length used in the integration ($h_1 = 4a/M$, where a is side of square contour and M is the number of points used to evaluate the integral). Starting from a value of $M = 16$ we have used the Romberg integration to improve on the accuracy by evaluating the integral for a series of values of M in geometric progression with ratio 2. The calculations are contined until two successive approximations to the integral differ by less than a prescribed small constant. It should be noted that the approximation to the derivative introduces odd powers of h_1 in the asymptotic error expansion which renders the Romberg integration somewhat ineffective. In order to obtain a faster convergence of the integral it would be desirable to compute the derivative of the determinant explicitly. But for obtaining very high accuracy the use of this method would not be economical and is, in fact, not necessary since once the eigenvalues are approximately located, the iterative methods can be very efficiently used to get any desired accuracy.

The number of eigenvalues of A will be at most $(N + 1)\,rm$, which is a finite number, while the actual problem has an infinite number of eigenvalues. Such a situation seems rather uncomfortable but as has been pointed out by Fox [3] and Keller [5], the lower eigenvalues are approximated quite accurately, while the higher eigenvalues are poorly represented. As will be seen in Section 3 the truncation error in approximating the eigenvalues is of same order as the local truncation error of the difference scheme, which for the central difference scheme considered here is $(h^2/24)(By''' - 3Cy'')$. Now in physical problems the higher eigenfunctions have a larger number of nodes which imply very large derivatives. For example, if the eigenfunctions are of form $\sin(nt)$, then the truncation error which is of the order of h^2n^3 will increase rapidly with n. For values of n of the order of the number of points in the difference scheme the error will be embarassingly large. This is to be expected since in general the nth-order eigenfunction will have n nodes, which can never be approximated by any

numerical method involving a number of points which are of same order or less than n.

For determining the eigenvector we use the inverse iteration method which is not directly applicable to the generalized eigenvalue problem. But if we have already determined λ_k to be the eigenvalue for the generalized eigenvalue problem, it can be easily shown that this method can be used to obtain the corresponding eigenvectors. In fact, it is found that in almost all cases we require only one iteration to obtain the eigenvector.

It should be noted that the reduction to a first order system of differential equations generates a rather large matrix as compared to that for the equivalent system of higher order differential equations. However the band-width of the matrix will increase with the order of the differential equations involved. Further it is found that in problems of hydrodynamic stability the elimination of variables to get higher order equations introduces singularities in the resulting equations and so the numerical treatment becomes difficult.

3. Higher-Order Accuracy

The simplest way for obtaining higher-order accuracy is probably the application of Richardson's deferred approach to the limit or as it is also termed the $h \to 0$ extrapolation which has been discussed by Fox [3] and Keller [5]. In this method the eigenvalue problem is solved for a series of values of N, and the errors involved are eliminated by assuming it to be a power series in h. For the eigenvalue problem this necessitates an essentially complete repetition of all the work for each value of N, and the process is thus very time consuming. An alternative method is to use a more accurate difference scheme, but this increases the bandwidth of the matrix obtained. In the present work we have used explicit evaluation of difference corrections as given below, which requires very little additional calculations to obtain the first-order correction, and which is sufficiently good for most purposes. However, it is found that to obtain still higher-order accuracy with this method is not free from round-off errors and may not be even possible.

By using the Taylor series we can write

$$L_h(\lambda)\, u_j = L(\lambda)\, u_j + T_h(\lambda)\, u_j\,, \tag{8}$$

where $T_h(\lambda)\, u_j$ gives the local truncation error at t_j. For the central difference scheme used here we can expand $T_h(\lambda)$ in a power series in terms of h^2:

$$T_h(\lambda) = T_1(\lambda) + T_2(\lambda) + \cdots, \tag{9}$$

where $T_i(\lambda)$ is $O(h^{2i})$. Similarly we can write the actual eigenvalue Λ and $y(t_j)$ in the form

$$\Lambda = \lambda_0 + \lambda_1 + \cdots, \qquad y(t_j) = u_{j0} + u_{j1} + \cdots, \tag{10}$$

using

$$U_i = (u_{0i}^T, u_{1i}^T, \ldots, u_{Ni}^T)^T.$$

Here U_0 is the eigenvector for the difference operator corresponding to the eigenalue λ_0. If V_0 is the eigenvector of the transposed matrix for the difference problem for the same eigenvalue, then

$$L_h(\lambda_0)\, U_0 = 0, \qquad L_h^T(\lambda_0)\, V_0 = 0 \qquad \text{or} \qquad V_0^T L_h(\lambda_0) = 0. \tag{11}$$

We shall obtain only the first-order corrections λ_1 and U_1 to the eigenvalue and eigenvector, respectively; that is, we shall find U_1 and λ_1 such that

$$L_h(\lambda_0 + \lambda_1)(U_0 + U_1) = T_1(\lambda_0)\, U_0\,. \tag{12}$$

Retaining only the first-order terms we obtain, using (11),

$$L_h(\lambda_0)\, U_1 + \lambda_1 L_h'(\lambda_0)\, U_0 = T_1(\lambda_0)\, U_0\,, \tag{13}$$

where

$$L_h' = \frac{\partial L_h}{\partial \lambda}\,.$$

Multiply Eq. (13) from the left by V_0^T and use Eq. (11) to get

$$\lambda_1 = \frac{V_0^T T_1(\lambda_0)\, U_0}{V_0^T L_h'(\lambda_0)\, U_0}\,.$$

Thus, we see that the first-order correction to the eigenvalue which gives an estimate of the truncation error is of same order as the local truncation error of the difference scheme, a result which has been rigorously proved by Keller [9] for the standard eigenvalue problem. However, to proceed beyond the first order we will require U_1. But it will not be possible to solve Eq. (13) for U_1 since $L_h(\lambda_0)$ is a singular matrix. Equation (12) can be solved for $(U_0 + U_1)$ but to calculate U_1 from that will probably involve very large round-off errors.

The calculation of λ_1 requires the operator $T_1(\lambda_0)$ and we outline a procedure to approximate it numerically. By using the Taylor series expansion about the point $t_{j-1/2}$, we get

$$L_h y = Ly + \frac{h^2}{24}(By''' - 3Cy'') + \frac{h^4}{1920}(By^{\mathrm{v}} - 5Cy^{\mathrm{iv}}) \tag{14}$$

which yields

$$T_1 y = \frac{h^2}{24}(By''' - 3Cy''),$$

where all quantities are evaluated at $\lambda = \lambda_0$, $t = t_{j-1/2}$.

The derivatives $y''(t_{j-1/2})$ and $y'''(t_{j-1/2})$ can be calculated by using the four-point central difference approximation. But if that is used in (14), then the additional error

introduced is roughly 10 times the error term $(h^5/1920)(By^{v} - 5Cy^{iv})$. Thus it would be better to use the six-point central difference approximation for y'' and y'''. At the boundaries, of course, the central difference cannot be used and the one-sided difference should be used. The use of the one-sided difference can be avoided by computing at points outside the range $[a, b]$. This dodge may be useful for the two-point boundary value problem, but it appears that for the eigenvalue problem it may not be any simpler than the use of the one-sided difference, especially when the number of equations is large.

The evaluation of eigenvectors requires very little calculation, as in the process of finding the eigenvalues the determinant is already evaluated at the required value of λ. The triangular decomposition obtained in that process can be effectively used to calculate both U_0 and V_0, without any additional computation of the coefficients, etc. Further, if the eigenvalues are obtained to a good accuracy, then it requires only a single iteration to get the eigenvectors. Only additional calculation is required to obtain the truncation error and the derivative matrix for the problem, which is roughly equivalent to evaluating the determinant for a single value of λ. Thus, the additional computation needed in this process is an order of magnitude less than that for evaluating the eigenvalues, which requires several iterations. Besides we obtain eigenvectors as a by-product of this calculation which are very useful in some physical problems.

4. Numerical Example and Discussion

We shall illustrate the working of the method presented above with a rather simple example, that of an isothermal fluid layer bounded by plane horizontal boundaries at $z = 0$ and $z = 1$. As can be seen from Eq. (4) if $\Gamma = 1$, the temperature T_0 is constant and we have an isothermal fluid layer. Further, for such a problem all the coefficients of the equation are constant and hence an exact analytical solution can be written. We have chosen rigid boundary conditions $\rho_0 W = 0$ at $z = 0$ and $z = 1$. The analytical solutions are of the form $\rho_0 W = \exp(-\frac{1}{2}z)\sin(m\pi z)$, where m is an integer. With different values of m we obtain different modes starting from F-mode($m = 1$) and the eigenvalues obtained by the above method can be compared with exact values obtained separately. The parameters for this problem are chosen as follows: $\gamma = 0.9$; $q = 0.05284$; $k = 1.000$.

Table I shows the eigenvalues obtained without applying the difference correction for various values of N (the number of intervals in the difference scheme). Table I gives the eigenvalues for the first four acoustic modes, namely, F, $P1$, $P2$, and $P3$. The first line gives the exact eigenvalues which can be compared with the calculated values. It can be seen that the error is roughly proportional to h^2m^3 or m^3/N^2. Table II lists the eigenvalues after applying difference corrections and it can be seen that the error is roughly proportional to h^4m^5 or m^5/N^4. The most remarkable feature of this computation is that for $N = 100$ the lowest eigenvalue is accurate up to eight digits.

TABLE 1

Acoustic Modes for Isothermal Atmosphere (without Difference Correction); $C_p/C_v = 0.900$, $q = 5.284\text{E}{-2}$, $k = 1.000$

	F-Mode ($m = 1$)		$P1$-Mode ($m = 2$)		$P2$-Mode ($m = 3$)		$P3$-Mode ($m = 4$)	
	λ_r	λ_i	λ_r	λ_i	λ_r	λ_i	λ_r	λ_i
Exact	2.902431E–3	3.1651158	2.932893E–3	6.0546378	2.934955E–3	9.0039083	2.935343E–3	11.968639
5	2.905878E–3	3.254749	2.933989E–3	6.962739	2.935402E–3	13.08930	2.935555E–3	29.20977
50	2.902467E–3	3.165968	2.932906E–3	6.062226	2.934962E–3	9.030071	2.935347E–3	12.03125
200	2.902433E–3	3.165169	2.932894E–3	6.055112	2.934956E–3	9.005538	2.935344E–3	11.97253
500	2.902430E–3	3.165124	2.932893E–3	6.054714	2.934955E–3	9.004169	2.935344E–3	11.96926
1000	2.902430E–3	3.165117	2.932892E–3	6.054656	2.934955E–3	9.003974	2.935342E–3	11.96880
2000	2.902431E–3	3.165114	2.932894E–3	6.054643	2.934955E–3	9.003924	2.935349E–3	11.96868
	Error $= (\lambda_N - \lambda_{\text{exact}}) \times 1.0\text{E}6$							
5	3.45	89633	1.10	908101	0.45	4085390	0.21	17241130
50	0.04	852	0.01	7588	0.01	26163		62610
200		53		474		1630		3890
500		8		76		261		620
1000		1		18		66		160
2000		−2		5		16		40

TABLE II

Acoustic Modes for Isothermal Atmosphere (with Difference Correction); $C_p/C_v = 0.900$, $q = 5.284\text{E}{-2}$, $k = 1.000$

	F-Mode ($m = 1$)		*P*1-Mode ($m = 2$)		*P*2-Mode ($m = 3$)		*P*3-Mode ($m = 4$)	
N	λ_r	λ_i	λ_r	λ_i	λ_r	λ_i	λ_r	λ_i
Exact	2.902431E–3	3.165115800	2.932893E–3	6.054637848	2.934955E–3	9.003908327	2.935343E–3	11.96863975
5	2.903305E–3	3.18299156	2.932969E–3	5.73676252				
10	2.902432E–3	3.16477168	2.932899E–3	6.04481608	2.934962E–3	8.93011653	2.935349E–3	11.5575137
20	2.902430E–3	3.16509381	2.932889E–3	6.05393976	2.934957E–3	8.99881172	2.935339E–3	11.9481279
50	2.902431E–3	3.16511527	2.932893E–3	6.05441949	2.934956E–3	9.00376880	2.935343E–3	11.9680556
100	2.902431E–3	3.16511577	2.932892E–3	6.05463660	2.934955E–3	9.00389957	2.935344E–3	11.9686028
	Error $= (\lambda_N - \lambda_{\text{exact}}) \times 1.0\text{E}8$							
5	87	1787576	8	−31787533				
10		−34412	1	−982177	1	−7379180	1	−41112610
20		−2199		−69809		−509661		−2051190
50		−53		−1836		−13953		−58420
100		−3		−125		−876		−3700

Here we have used only single precision complex arithmetic on DEC-10 computer which gives an accuracy of roughly 1 part in 10^8. This clearly demonstrates that the computations involved are free from round-off errors. The eigenvectors obtained in this process are also compared with exact values. When the eigenvectors are normalized to get a maximum value of one, the maximum error for the F-mode is approximately 6×10^{-5} for $N = 100$ and increases to roughly 10^{-3} for the $P3$-mode. For $N = 50$ the errors are higher by a factor of 2–4.

This equation also has a series of real eigenvalues known as the convective modes. The interesting feature about these eigenvalues is that they have a limit point at $\lambda = 0$ and so a denumerably infinite number of eigenvalues are concentrated in a small region around $\lambda = 0$. It is found that as long as we are reasonably away from the limit point no serious problem is caused by it. To illustrate this we have used the method to locate eigenvalues in the region around the positive real axis and the results shown in Table III are obtained by using 51 mesh points ($N = 50$). The second and third

TABLE III

Convective Modes for Isothermal Atmosphere;
$C_p/C_v = 0.900$, $q = 5.284\text{E}{-2}$, $k = 1.000$

m	a	b	c	d	e
1	7.4777205E–2	5.0394160E–8	7.4775927E–2	7.4801782E–2	7.480177045E–2
2	3.0487608E–2	−6.0187963E–7	3.0502161E–2	3.0559207E–2	3.055914993E–2
3	1.6369830E–2	1.2198290E–6	1.6340423E–2	1.6418595E–2	1.641845333E–2
4	1.0046688E–2	−3.1363919E–5	1.0047383E–2	1.0139474E–2	1.013923383E–2
5	6.7387404E–3	8.8413436E–5	6.7345953E–3	6.8358423E–3	6.835505908E–3
6	4.7885919E–3	−5.7117014E–5	4.7919952E–3	4.8993522E–3	4.898939746E–3
7	3.5641736E–3	−5.8921154E–7	3.5622970E–3	3.6738012E–3	3.673346992E–3
8	2.7331980E–3	1.4461245E–6	2.7377670E–3	2.8521128E–3	2.851666494E–3
9	2.1621888E–3	−8.5610199E–7	2.1594956E–3	2.2757756E–3	2.275401970E–3
10	1.7399815E–3	−1.4326302E–9	1.7390473E–3	1.8565983E–3	1.856377146E–3

[a,b] Real and imaginary parts of eigenvalues as located by the scarch routine.
[c] Real eigenvalues as refined by secant iteration.
[d] Real eigenvalues after applying difference correction.
[e] Exact value of the eigenvalues.

columns give the real and imaginary parts of the eigenvalues as located by the search routine using 512 points for evaluating the integrals. The search routine is used only for illustration, as it is clear that in this case eigenvalues can be located much more efficiently by just looking for sign changes. The fourth column contains the same real eigenvalues as refined by the secant iteration method which is used to improve on the values located earlier. The fifth column gives the same eigenvalues after applying the difference correction. It can be seen that despite the proximity of the limit point the search routine gives a reasonable accuracy of about three or four significant

digts, which is more than enough for the iterative methods to converge very fast to any desired accuracy permissible by the round-off errors. It may also be seen that the difference correction is very effective in improving the accuracy of eigenvalues. The corrected eigenvalues have an accuracy of better than 1 part in 10^4 even for the tenth eigenvalue which is very close to the limit point.

For a standard eigenvalue problem of the form $Ly = \lambda y$, where L is a linear differential operator and y is a scalar variable, it is known that there are at most a denumerably infinite number of eigenvalues with no finite limit point. For the generalized eigenvalue problem again the number of eigenvalues will be at most denumerably infinite but in general they can have a finite limit point. This is true for most of the generalized eigenvalue problems which we encounter in hydrodynamic stability analysis. Thus, for example, for the isothermal fluid layer without dissipation ($q = 0$ in (3)) one of the series of eigenvalues is given by

$$\lambda_m^2 = \frac{(k^2 + \frac{1}{4} + m^2\pi^2)\,\gamma}{2}\left[-1 + \left(1 + \frac{4k^2(1-\gamma)}{\gamma^2(k^2 + \frac{1}{4} + m^2\pi^2)^2}\right)^{1/2}\right].$$

Clearly $\lambda \to 0$ as $m \to \infty$ and it is evident that the series has a limit point at $\lambda = 0$. In such cases there can be an infinite number of eigenvalues in a finite region of the complex plane and the method described above will run into difficulty if the region to be searched contains, or passes very close to, the limit point. Of course, the finite-difference operator will always have a finite number of eigenvalues but the nearer we approach the limit point the determination of the eigenvalues will become more and more unreliable because of high truncation errors. As we have already demonstrated even in such cases the eigenvalues computed are fairly reliable as long as the order of eigenvalues computed is reasonably small compared to the number of mesh points used in the difference scheme.

Admittedly the illustration used here is for an equation with constant coefficients, but we have applied this method successfully for equations with variable coefficients and the results are in perfect agreement with those obtained by others. To give an example we have used this method for determining all three eigenvalues corresponding to the symmetric stream function of the Orr–Sommerfeld equation for the plane Poiseuille flow (cf. Antar [2]) in the square $0 \leqslant \lambda_r \leqslant 0.4$, $-0.4 \leqslant \lambda_i \leqslant 0.0$. This equation provides a standard eigenvalue problem for a fourth-order differential equation. Recently, this problem has been solved numerically by a number of workers with whom it is possible to compare our results. We have compared our results with Mack [10], who has tabulated 32 known eigenvalues of this equation for symmetric stream functions and for disturbance wave number $\alpha = 1$, and the Reynolds number $R = 10{,}000$, which lie in the rectangle $0 \leqslant \lambda_r \leqslant 1.0$, $-1.1 \leqslant \lambda_i \leqslant 0$. The results are displayed in Table IV. Here, we have used 101 mesh points ($N = 100$) and the search routine used 512 points to evaluate the integrals. It can be seen that the results are in very good agreement with Mack [10] and despite the extremely high value of the Reynolds number R, it requires only 101 mesh points to get an accuracy of 10^{-5}.

The method described in this paper gives a very practical approach for calculating eigenvalues and eigenvectors for differential operators. On an average it takes about

TABLE IV

Eigenvalues for Orr–Sommerfeld Equation of Plane Poiseuille Flow;
Wave Number = 1.000; Reynolds Number = 10000

a	b	c	d	e	f	g	h
0.192413	−0.182600	0.192172	−0.182541	0.190054	−0.182810	0.19006	−0.18282
0.351707	0.121757	0.351991	−0.121705	0.349098	−0.124498	0.34911	−0.12450
0.373708	−0.235530	0.373628	−0.235704	0.368463	−0.238777	0.36850	−0.23882

[a,b] Real and imaginary parts of eigenvalues as located by the search routine.
[c,d] Real and imaginary parts of eigenvalues as refined by secant iteration.
[e,f] Real and imaginary parts of eigenvalues after applying difference correction.
[g,h] Real and imaginary parts of eigenvalues as tabulated by Mack [10].

1–2 sec. of CPU time on a DEC-10 computer to compute one complex eigenvalue for Eq. (3) by using iterative methods with 51 mesh points, while the search routine requires about 10–20 times more CPU time. Thus, it is clear that the search routine is useful only when the usual methods have failed. It is found that for integrals in the search routine to converge the size of square contour should be sufficiently small; the actual limit depending on the equation under consideration. As an example, the three eigenvalues of the Orr–Sommerfeld equation were determined by using the square contour with side 0.4, while for magnetohydrodynamic slow modes it is found that the size of square has to be less than about 0.1. Despite these limitations the search routine has to be used in such cases since other methods fail completely. Under such circumstances the iteration fails to converge even if the starting value differs by only 0.01 from the actual eigenvalue.

However, a word of caution is necessary since in this method we actually locate the eigenvalues of the finite-difference operator rather than that of the original differential operator. Hence, if an attempt is made to locate the higher-order eigenvalues which are not very accurately represented by the difference operator, the results would be obviously unreliable. We expect this method to be applicable to even more generalized class of eigenvalue problems, where the coefficients of the differential equations are, in general, analytic functions of λ. There is, of course, no question of using this method for locating eigenvalues in an infinite region of the complex plane, which would be of great interest to physicists, especially for settling the question of the existence of eigenvalues in the complex half-plane.

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